Int J Theor Phys (2010) 49: 549–563
DOI 10.1007/s10773-009-0236-2

Nonlinear Gravitational Waves: Their Form and Effects

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Received: 6 July 2009 / Accepted: 22 December 2009 / Published online: 5 January 2010

Abstract A gravitational wave must be nonlinear to be able to transport its own source, that is, energy and momentum. A physical gravitational wave, therefore, cannot be represented by a solution to a linear wave equation. Relying on this property, the second-order solution describing such physical waves is obtained. The effects they produce on free particles are found to consist of nonlinear oscillations along the direction of propagation.

Keywords Gravitational waves · Nonlinear gravitational waves

1 Introduction

For a long period after the advent of general relativity, the question of the existence or not of gravitational waves was a very controversial issue. In the seventies, the discovery of a binary pulsar system whose orbital period changes according to the predicted wave emission put an end to the controversy [1]. In fact, that discovery provided an indirect but compelling experimental evidence for the existence of gravitational waves [2–5]. That evidence, however, did not provide any clue on their form and effects. As a matter of fact, although widely considered a finished topic [6], it actually remains plagued by many obscure points [7, 8]. For example, although there seems to be an agreement that the transport of energy-momentum by gravitational waves is essentially a nonlinear phenomenon, instead of going to the second order, one usually assumes a "mixed" procedure, which consists basically in assuming that gravitational waves carry energy (are nonlinear, or at least second order), but at the same time, because this energy is very small, one also assumes that its evolution can approximately be described by a linear (first order) equation [9]. When one speaks of "linear

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gravitational waves", therefore, one means nonlinear gravitational waves whose dynamics is assumed to be approximately described by a linearized equation. This means that, in addition to the sequential levels of accuracy implied by the perturbative analysis, there is also another approximation, implied by the "mixed" approach, according to which all first-order equations describing a gravitational wave are to be interpreted as only nearly correct [10].

Such assumption, however, is an unjustified surmise: the issue is not a matter of approximation, but a conceptual question.[1] A gravitational wave either does or does not carry energy: if it carries, no matter how small it is, it cannot satisfy a linear equation. It is, therefore, conceptually unsatisfactory to assume that a gravitational wave satisfying a linear equation is able to transport energy and momentum. If applied to a Yang–Mills propagating field [12], the approximation described above would correspond to assume that, for a field with small-enough amplitude, its evolution can be accurately described by a linear equation. Of course, this is plainly wrong: a Yang–Mills propagating field must necessarily be nonlinear to carry its own (color) source, otherwise it is not a Yang–Mills field. Analogously, a gravitational wave must necessarily be nonlinear to transport its own source—that is, energy and momentum.

Taking into account these premises, a critical review of the gravitational wave theory has been published recently [13]. In that paper, it was discussed why the standard approach to the gravitational wave theory is not satisfactory. Here, instead of using the mixed approach, we proceed to the second order and obtain the corresponding nonlinear gravitational wave. It is important to remark that this re-interpretation of the gravitational wave concept has no implications for the usual expressions of the power emitted by a mechanical source. In particular, the (nonlinear) quadrupole radiation formula gives a correct account of the energy emitted by a binary pulsar, for example. The only we claim is that the energy and momentum are not transported away by linear, but by nonlinear waves. The basic purpose of the present paper is to make an analysis of these nonlinear waves, as well as of their effects on test particles.

2 Linear Approximation

2.1 Linear Wave Equation

The study of gravitational waves involves basically the weak field approximation of Einstein equation

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = \frac{8\pi G}{c^4}\, \Theta_{\mu\nu}, \tag{1}$$

where $\Theta_{\mu\nu}$ is the source energy-momentum tensor. That is arrived at by expanding the metric tensor according to

$$\sqrt{-g}\, g^{\mu\nu} = \eta^{\mu\nu} + \varepsilon\, h^{\mu\nu}_{(1)} + \varepsilon^2\, h^{\mu\nu}_{(2)} + \dots, \tag{2}$$

where ε is a small parameter introduced to label the successive orders in this perturbation scheme. When the metric tensor is expanded according to (2), we are automatically assuming that there is a background Minkowskian structure in spacetime, with metric $\eta_{\mu\nu}$. Accordingly, the gravitational waves are interpreted as perturbations

$$h^{\mu\nu} = \varepsilon\, h^{\mu\nu}_{(1)} + \varepsilon^2\, h^{\mu\nu}_{(2)} + \dots \tag{3}$$

[1] There are other arguments against this assumption. See, for example, [11].

propagating on that fixed Minkowskian background. This interpretation is consistent with general relativity, as well as with the point of view of field theory, according to which a field always propagates on a background spacetime [14].

Assuming expansion (2), the first order Ricci tensor is

$$R_{(1)\mu\nu} = \partial_\lambda \Gamma^\lambda_{(1)\mu\nu} - \partial_\nu \Gamma^\lambda_{(1)\mu\lambda}. \tag{4}$$

Using the first order Christoffel connection

$$\Gamma^\lambda_{(1)\mu\nu} = \frac{1}{2}\left(-\partial_\mu h^\lambda_{(1)\nu} - \partial_\nu h^\lambda_{(1)\mu} + \partial^\lambda h_{(1)\mu\nu} + \frac{1}{2}\delta^\lambda_\nu\, \partial_\mu h_{(1)} + \frac{1}{2}\delta^\lambda_\mu\, \partial_\nu h_{(1)} - \frac{1}{2}\eta_{\mu\nu}\, \partial^\lambda h_{(1)}\right), \tag{5}$$

the Ricci tensor and the scalar curvature become, respectively,

$$R_{(1)\mu\nu} = \frac{1}{2}\left(\Box h_{(1)\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h_{(1)} - \partial_\lambda \partial_\mu h^\lambda_{(1)\nu} - \partial_\nu \partial^\lambda h_{(1)\mu\lambda}\right) \tag{6}$$

and

$$R_{(1)} = -\frac{1}{2}\Box h_{(1)} - \partial_\lambda \partial_\rho h^{\lambda\rho}_{(1)}, \tag{7}$$

where $\Box = \eta^{\rho\lambda}\, \partial_\rho \partial_\lambda$ is the flat spacetime d'Alembertian, and $h_{(1)} = h^\lambda_{(1)\lambda}$. In consequence, the first order sourceless gravitational field equation becomes

$$\Box h_{(1)\mu\nu} - \partial_\lambda \partial_\mu h^\lambda_{(1)\nu} - \partial_\nu \partial^\lambda h_{(1)\mu\lambda} + \eta_{\mu\nu} \partial_\lambda \partial_\rho h^{\lambda\rho}_{(1)} = 0. \tag{8}$$

Now, as is well known, wave equation (8) is invariant under (infinitesimal) general spacetime coordinate transformations. Analogously to the electromagnetic wave equation, which is invariant under gauge transformations, the ambiguity of the gravitational wave equation can be removed by choosing a particular class of coordinate systems—or gauge, as it is usually called. The most convenient choice is the class of harmonic coordinate systems, which at first order is fixed by

$$\partial_\mu h^{\mu\nu}_{(1)} = 0. \tag{9}$$

In this case, the field equation (8) reduces to the relativistic wave equation

$$\Box\, h^\rho_{(1)\nu} = 0. \tag{10}$$

2.2 Linear Waves

A monochromatic plane-wave solution to (10) has the form

$$h_{(1)\mu\nu} = A_{(1)\mu\nu} \exp[i k_\rho x^\rho], \tag{11}$$

where $A_{(1)\mu\nu} = A_{(1)\nu\mu}$ is the polarization tensor, and the wave vector k^ρ satisfies

$$k_\rho\, k^\rho = 0. \tag{12}$$

The harmonic coordinate condition (9), on the other hand, implies

$$k_\mu\, h^\mu_{(1)\nu} = 0. \tag{13}$$

In analogy with the Lorentz gauge in electromagnetism, it is possible to further specialize the harmonic class of coordinates to a particular coordinate system. Once this is done, the coordinate system becomes completely specified, and the components $A_{(1)\mu\nu}$ turn out to represent only physical degrees of freedom. A quite convenient choice is the so called *transverse–traceless* coordinate system (or gauge), in which [15]

$$h^{\rho}_{(1)\rho} = 0 \quad \text{and} \quad h^{\mu}_{(1)\nu}\, U^{\nu}_{(0)} = 0, \tag{14}$$

with $U^{\nu}_{(0)}$ an arbitrary, constant four-velocity.

Now, although the coordinate system $\{x^{\mu}\}$ has already been completely specified (the transverse-traceless coordinate system), we still have the freedom to choose different local Lorentz frames e^a. Since the metric $g_{\mu\nu} = \eta_{ab}\, e^{a}{}_{\mu} e^{b}{}_{\nu}$ is invariant under changes of frames, the metric perturbation will also be invariant. In particular, it is always possible to choose a specific frame, called *proper frame*, in which $U^{\nu}_{(0)} = \delta_0{}^{\nu}$. In this frame, as can be seen from the second of (14),

$$h^{\mu}_{(1)0} = 0 \tag{15}$$

for all μ. Linear waves satisfying these conditions are usually assumed to represent a plane gravitational wave in the transverse-traceless gauge, propagating in the vacuum with the speed of light. Its physical significance, however, can only be determined by analyzing the energy and momentum it transports.

2.3 Energy and Momentum Transported by Linear Waves

The energy-momentum tensor of any matter (or source) field ψ is proportional to the functional derivative of the corresponding Lagrangian with respect to the spacetime metric. Since such a derivative does not change the order of the Lagrangian in the matter field ψ, both the Lagrangian and the energy-momentum tensor will be of the same order in the field variable ψ. For example, both Maxwell's Lagrangian and its corresponding energy-momentum tensor are quadratic in the electromagnetic field. Now, it is a well known fact that the gravitational field is itself a source of gravitation. This means that the gravitational energy-momentum current should appear explicitly in the gravitational field equation. Accordingly, the wave equation (10) should read

$$\Box\, h_{(1)\mu\nu} = \frac{16\pi G}{c^4}\, t_{(1)\mu\nu}. \tag{16}$$

At the *linear approximation*, therefore, the gravitational energy–momentum density $t_{(1)\mu\nu}$ is restricted to be linear. However, since the energy–momentum density is at least quadratic in the field variable, $t_{(1)\mu\nu}$ vanishes in the linear approximation, leading to the wave equation (10).

The above property is a crucial difference between linear gravity and electromagnetism, and is often a source of confusion. Even though the electromagnetic waves are linear, they do transport energy and momentum. There is no any inconsistency in this result because neither energy nor momentum are sources of electromagnetic field, and consequently the energy-momentum tensor does not appear explicitly in the electromagnetic field equation. In other words, even though the electromagnetic field equations are linear, the energy-momentum tensor is not restricted to be linear. The linearity of the electromagnetic wave equation, however, restricts the electromagnetic self–current to be linear, and consequently to vanish. This means that the electromagnetic wave is unable to transport its own source, that is, electric charge. A linear gravitational wave is similarly unable to transport its own source, that is, energy and momentum. Only a nonlinear wave will be able to do it. This a subtle, but fundamental difference between electromagnetic and gravitational waves.

The consistency of this result can be verified by analyzing the generation of linear waves. In the presence of a source, the first order field equation reads

$$\Box h_{(1)\mu\nu} = \frac{16\pi G}{c^4}\,\Theta_{(1)\mu\nu}, \tag{17}$$

with $\Theta_{(1)\mu\nu}$ the first order source energy-momentum tensor. As a consequence of the coordinate condition (9), it is easy to see that

$$\partial^\mu \Theta_{(1)\mu\nu} = 0. \tag{18}$$

Instead of the usual covariant derivative, $\Theta_{(1)\mu\nu}$ is conserved with an ordinary derivative at the first order. Since this is a true conservation law, in the sense that it leads to a time conserved *charge*, we can conclude that in the linear approximation a mechanical system cannot lose energy in the form of gravitational waves.[2] As discussed in Sect. 1, this problem is usually circumvented by assuming the mixed approach, according to which this equation is to be interpreted as nearly true.

2.4 Generation of Linear Waves

Let us consider the first order field equation (17). A solution is the retarded potential

$$h_{(1)\mu\nu} = \frac{4G}{c^4}\int \frac{d^3x'}{|\vec{x}-\vec{x}\,'|}\,\Theta_{(1)\mu\nu}(t',\vec{x}\,'), \tag{19}$$

with the source considered in the retarded time

$$t' = t - \frac{|\vec{x}-\vec{x}\,'|}{c}. \tag{20}$$

At large distances from the source we can expand

$$|\vec{x}-\vec{x}\,'| \simeq r - \vec{x}\,'\cdot\hat{n} + \dots, \tag{21}$$

where $r = |\vec{x}|$ is the distance from the source, and $\hat{n}$ is a unit vector in the direction of $\vec{x}$. The leading order term of $h_{(1)\mu\nu}$ is obtained by replacing $|\vec{x}-\vec{x}\,'|$ in the denominator of (19) with r,

$$h_{(1)\mu\nu} = \frac{4G}{rc^4}\int d^3x'\,\Theta_{(1)\mu\nu}(t',\vec{x}\,'), \tag{22}$$

where now

$$t' = t - \frac{r}{c} + \frac{\vec{x}\,'\cdot\hat{n}}{c}. \tag{23}$$

[2] This is consistent with the fact that linear gravitational waves do not transport energy nor momentum. The existence of a linear solution is a mere consequence of the use of a perturbative scheme, but alone it does not represent the physical wave.

The Fourier transform of $\Theta_{(1)\mu\nu}$ is

$$\Theta_{(1)\mu\nu}(t', \vec{x}\,') = \int \frac{d^4k}{(2\pi)^4}\, \tilde{\Theta}_{(1)\mu\nu}(\omega, \vec{k})\, e^{-i\omega t' + i\vec{k}\cdot\vec{x}'}. \tag{24}$$

Substituting in (22) and performing the integrations in d^3x' and d^3k, we obtain [2]

$$h_{(1)\mu\nu} = \frac{4G}{rc^4} \int_0^\infty \frac{dw}{2\pi}\, \tilde{\Theta}_{(1)\mu\nu}(\omega, \omega\hat{n}/c) e^{-i\omega(t-r/c)}. \tag{25}$$

We see from this expression that, if the source oscillates with a single frequency ω, the plane wave $h_{(1)\mu\nu}$ will necessarily propagate with the same frequency. However, we know from the quadrupole radiation formula that, if the source oscillates with frequency ω, the gravitational radiation should come out with frequency 2ω.[3] The reason for this factor of 2 is that both the generation and the effects of gravitational waves on free particles are essentially tidal effects, which we know to occur twice during a complete cycle. This is a clear indication that $h_{(1)\mu\nu}$ alone cannot represent the physical gravitational wave.

3 Second Order Approximation

3.1 Second–Order Wave Equation

At the second order of the iterated perturbation scheme, the harmonic coordinate condition reads [14]

$$\partial_\mu h^\mu_{(2)\nu} = 0. \tag{26}$$

In these coordinates, the second order gravitational field equation can be written in the form

$$\Box\, h^\rho_{(2)\nu} = \frac{16\pi G}{c^4} \left(t^\rho_{(2)\nu} + \Theta^\rho_{(2)\nu} \right), \tag{27}$$

where $t^\rho_{(2)\nu} \equiv t^\rho_{(2)\nu}(h_{(1)}, h_{(1)})$ represents all terms coming from the left-hand side of Einstein equation, in addition to the d'Alembertian term. It can be interpreted as the second order energy-momentum pseudotensor of the gravitational field [9].

Far away from the sources, the second order gravitational waves are governed by the sourceless version of the wave equation (27),

$$\Box\, h^{\mu\nu}_{(2)} = \frac{16\pi G}{c^4}\, t^{\mu\nu}_{(2)} \equiv N^{\mu\nu}(h_{(1)}, h_{(1)}), \tag{28}$$

where, already considering the traceless gauge condition $h_{(1)} = 0$,

$$\begin{aligned} N^{\mu\nu}(h_{(1)}, h_{(1)}) = {} & -h^{\rho\sigma}_{(1)} \partial_\rho \partial_\sigma h^{\mu\nu}_{(1)} + \frac{1}{2} \partial^\mu h_{(1)\rho\sigma} \partial^\nu h^{\rho\sigma}_{(1)} + \partial_\sigma h^{\mu\rho}_{(1)} (\partial^\sigma h^\nu_{(1)\rho} + \partial_\rho h^{\nu\sigma}_{(1)}) \\ & - \partial^\mu h_{(1)\rho\sigma} \partial^\rho h^{\nu\sigma}_{(1)} - \partial^\nu h_{(1)\rho\sigma} \partial^\rho h^{\mu\sigma}_{(1)} \\ & + \frac{\eta^{\mu\nu}}{2} \left(\partial_\rho h_{(1)\sigma\tau} \partial^\sigma h^{\rho\tau}_{(1)} - \frac{1}{2} \partial_\tau h_{(1)\rho\sigma} \partial^\tau h^{\rho\sigma}_{(1)} \right). \end{aligned} \tag{29}$$

[3] See, for example, [2, p. 105].

Using for $h^{\mu\nu}_{(1)}$ the plane wave solution (11), the wave equation becomes

$$\Box h^{\mu\nu}_{(2)} = \Big[k_\rho k_\sigma A^{\rho\sigma}_{(1)} A^{\mu\nu}_{(1)} - \frac{1}{2} k^\mu k^\nu A_{(1)\rho\sigma} A^{\rho\sigma}_{(1)} - k_\sigma k^\sigma A^{\mu\rho}_{(1)} A^{\nu}_{(1)\rho} - k_\sigma k_\rho A^{\mu\rho}_{(1)} A^{\nu\sigma}_{(1)} + k^\nu k^\rho A_{(1)\rho\sigma} A^{\mu\sigma}_{(1)} + k^\mu k^\rho A_{(1)\rho\sigma} A^{\nu\sigma}_{(1)} - \frac{\eta^{\mu\nu}}{2} \Big(k_\rho k^\sigma A_{(1)\sigma\tau} A^{\rho\tau}_{(1)} - \frac{1}{2} k_\tau k^\tau A_{(1)\rho\sigma} A^{\rho\sigma}_{(1)} \Big) \Big] \exp[i 2 k_\rho x^\rho]. \tag{30}$$

Use of the constraints (12) and (13) reduces it to

$$\Box h^{\mu\nu}_{(2)} = -\frac{\Phi_{(2)}}{2} k^\mu k^\nu \exp[i\, 2\, k_\rho x^\rho], \tag{31}$$

with

$$\Phi_{(2)} = A_{(1)\rho\sigma} A^{\rho\sigma}_{(1)}. \tag{32}$$

It is worth mentioning that the second-order wave equation is quadratic in the first-order solution $h^{\mu}_{(1)\nu}$. The factor “2” in the exponential of the right-hand side is a reminder of this nonlinear, quadratic dependence.

3.2 Second-Order Nonlinear Waves

A general solution to the wave equation (31) is given by a solution to the homogeneous equation plus a particular solution to the non-homogeneous equation. A monochromatic traveling-wave solution can then be written in the form

$$h^{\mu\nu}_{(2)} = \big(A^{\mu\nu}_{(2)} + i B^{\mu\nu}_{(2)} \big) \exp[i\, 2 k_\rho x^\rho], \tag{33}$$

where

$$A^{\mu\nu}_{(2)} = -\frac{\Phi_{(2)}}{16} \eta^{\mu\nu} \tag{34}$$

and

$$B^{\mu\nu}_{(2)} = \frac{\Phi_{(2)}}{8} \frac{K_\theta x^\theta}{K_\sigma k^\sigma} k^\mu k^\nu, \tag{35}$$

with K_α an arbitrary wave number four-vector. As a direct inspection shows, this solution satisfies the harmonic coordinate condition (26). The physical gravitational wave is represented by the real part of the solution, that is,

$$h^{\mu\nu}_{(2)} = A^{\mu\nu}_{(2)} \cos[2 k_\rho x^\rho] - B^{\mu\nu}_{(2)} \sin[2 k_\rho x^\rho]. \tag{36}$$

Observe that the amplitude $B^{\mu\nu}_{(2)}$ depends explicitly on the wave number—or equivalently, on the frequency of the wave. This is a typical property of nonlinear waves.

The amplitude of the first part of the solution satisfies

$$A^{\mu}_{(2)\mu} \equiv A_{(2)} = -\frac{\Phi_{(2)}}{4} \quad \text{and} \quad k_\mu A^{\mu\nu}_{(2)} = \frac{1}{4} k^\nu A_{(2)}. \tag{37}$$

We consider now a laboratory frame—with a Cartesian coordinate system—from which the wave will be observed. In this case, only the diagonal components of $A^{\mu\nu}_{(2)}$ are non-vanishing and obey

$$A^{xx} = A^{yy} = A^{zz} = -A^{tt}. \tag{38}$$

More specifically,

$$\left(A^{\mu\nu}_{(2)}\right) = -\frac{\Phi_{(2)}}{16}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{39}$$

The second part, on the other hand, satisfies

$$B^{\mu}_{(2)\mu} \equiv B_{(2)} = 0 \quad \text{and} \quad k_\mu B^{\mu\nu}_{(2)} = 0. \tag{40}$$

If we consider, for example, a wave traveling in the z-direction of the Cartesian system, for which

$$k^\rho = (\omega/c, 0, 0, \omega/c)\,, \tag{41}$$

the coefficient $B^{\mu\nu}_{(2)}$ will be of the form

$$\left(B^{\mu\nu}_{(2)}\right) = \frac{\Phi_{(2)}\, K_\theta x^\theta\, \omega^2}{8 K_\alpha k^\alpha c^2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{42}$$

Considering both parts of the solution we see that the second order wave is neither transverse nor traceless.

As already seen, if r denotes the distance from the source, the amplitude of the first order solution scales according to $A^{\mu\nu}_{(1)} \sim 1/r$. As an immediate consequence, $\Phi_{(2)} \sim 1/r^2$. This means that the amplitude of the first part of the solution (36) falls off as

$$A^{\mu\nu}_{(2)} \sim 1/r^2. \tag{43}$$

Due to an additional linear dependence on the distance, the amplitude of the second part falls off as

$$B^{\mu\nu}_{(2)} \sim 1/r. \tag{44}$$

At large distances from the source, therefore, the dominant solution will be of the form

$$h^{\mu\nu}_{(2)} \simeq B^{\mu\nu}_{(2)} \sin[2k_\rho x^\rho]. \tag{45}$$

Usually, second-order effects are supposed to fall off as $1/r^2$, and for this reason they are assumed to be neglectful at large distances from the source [15]. However, as shown above, the second-order gravitational wave $h^{\mu\nu}_{(2)}$ falls off as $1/r$, and consequently the arguments used to neglect them are not valid in this case. Observe also that, if the source oscillates with a single frequency ω, the field $h^{\mu\nu}_{(2)}$ will propagate, as appropriate for a quadrupole radiation, with a frequency 2ω [2]. This factor of 2 is a direct consequence of the nonlinear nature of the gravitational wave (see the comment just below (32)), and provides one more evidence that $h^{\mu\nu}_{(2)}$—and not $h^{\mu\nu}_{(1)}$—represents the physical (quadrupole) gravitational wave.

3.3 Generation of Nonlinear Waves

As can be seen from (26) and (27), the second order total energy-momentum tensor is conserved:

$$\partial_\mu \left[t^{\mu}_{(2)\nu} + \Theta^{\mu}_{(2)\nu}\right] = 0. \tag{46}$$

The source energy-momentum tensor, on the other hand, as determined by the second order Bianchi identity, is conserved only in the covariant sense:

$$\nabla_\mu \Theta^{\mu}_{(2)\nu} \equiv \partial_\mu \Theta^{\mu}_{(2)\nu} + \Gamma^{\mu}_{(1)\rho\mu}\, \Theta^{\rho}_{(1)\nu} - \Gamma^{\rho}_{(1)\nu\mu}\, \Theta^{\mu}_{(1)\rho} = 0. \tag{47}$$

At the second order, therefore, the source energy-momentum tensor is not truly conserved—it does not lead to a conserved *charge*. As a matter of fact, the above covariant conservation law is not a true conservation law, but simply an identity governing the exchange of energy and momentum between gravitation and matter [20]. As a consequence, in contrast to what happens at the first order, at the second order a mechanical system can lose energy in the form of gravitational waves.

It is important to remark once more that the usual expressions of the power emitted by a mechanical source, and in particular the quadrupole radiation formula, give a correct account of the energy emitted by a mechanical system. The reason is that nonlinear methods have always been used in the study of wave generation by such systems. Furthermore, the quadratic energy-momentum pseudotensor $t^{\rho}_{(2)\nu}$ is the complex traditionally used to calculate the energy and momentum transported by gravitational waves. What we claim here is that, instead of being transported by the linear waves $h^{\mu\nu}_{(1)}$, this energy is actually transported by the second-order gravitational wave $h^{\mu\nu}_{(2)}$. Notice from (27) that $t^{\mu\nu}_{(2)}$ appears as source of the second-order gravitational field $h^{\mu\nu}_{(2)}$. It represents, therefore, the energy and momentum transported by the second-order gravitational waves.

4 Effects on Free Particles

4.1 The Geodesic Deviation Equation

Let us consider, as usual, two nearby particles separated by the four-vector ξ^α. This vector obeys the geodesic deviation equation

$$\nabla_U\, \nabla_U \xi^\alpha = R^\alpha{}_{\mu\nu\beta}\, U^\mu\, U^\nu\, \xi^\beta, \tag{48}$$

where $U^\mu = dx^\mu/ds$, with $ds = g_{\mu\nu}\, dx^\mu dx^\nu$, is the four-velocity of the particles. Now, each order of the gravitational field expansion

$$R^\alpha{}_{\mu\nu\beta} = \varepsilon\, R^\alpha_{(1)\mu\nu\beta} + \varepsilon^2\, R^\alpha_{(2)\mu\nu\beta} + \dots, \tag{49}$$

which follows naturally from (2), will give rise to a different contribution to ξ^α. For consistence reasons, therefore, this vector must also be expanded:

$$\xi^\alpha = \xi^\alpha_{(0)} + \varepsilon\, \xi^\alpha_{(1)} + \varepsilon^2\, \xi^\alpha_{(2)} + \dots. \tag{50}$$

In this expansion, $\xi^\alpha_{(0)}$ represents the initial, that is, undisturbed separation between the particles. As the four-velocity U^μ depends on the gravitational field, it should also be expanded.

However, since the gravitational wave is interpreted as a perturbation of the flat Minkowski spacetime, the movement produced on free particles will also be considered in Minkowski spacetime. This means that we can write $U^\mu = U^\mu_{(0)} = dx^\mu/ds_{(0)}$, where

$$ds^2_{(0)} = \eta_{\mu\nu}\, dx^\mu dx^\nu \tag{51}$$

is the flat spacetime quadratic interval. Of course, the four-velocity $U^\mu_{(0)}$ depends on the choice of the initial condition—or equivalently, on the choice of the local Lorentz frame from which the phenomenon will be observed and measured. Using then the freedom to choose this frame (see Sect. 2.2), we can choose a frame fixed at one of the particles—called *proper frame*. In that frame, the proper time $s_{(0)}$ coincides with the coordinate x^0 [10], and the particle four–velocity assumes the form

$$U^\mu_{(0)} \equiv \delta^\mu{}_0 = (1, 0, 0, 0). \tag{52}$$

4.2 First-Order Effects

Considering that $\xi^\alpha_{(0)}$ represents simply the undisturbed separation between the particles, at the lowest order the geodesic deviation equation is

$$\frac{d^2\xi^\alpha_{(1)}}{ds^2_{(0)}} + U^\rho_{(0)}\, \partial_\rho \left(\Gamma^\alpha_{(1)\beta\gamma}\, U^\gamma_{(0)}\right) \xi^\beta_{(0)} = R^\alpha_{(1)\mu\nu\beta}\, U^\mu_{(0)}\, U^\nu_{(0)}\, \xi^\beta_{(0)}. \tag{53}$$

Substituting $U^\mu_{(0)}$ as given by (52), we get

$$\frac{d^2\xi^\alpha_{(1)}}{ds^2_{(0)}} + \partial_0 \Gamma^\alpha_{(1)\beta 0}\, \xi^\beta_{(0)} = R^\alpha_{(1)00\beta}\, \xi^\beta_{(0)}. \tag{54}$$

Using then the first order Riemann tensor

$$R^\alpha_{(1)\mu\nu\beta} = \partial_\nu \Gamma^\alpha_{(1)\mu\beta} - \partial_\beta \Gamma^\alpha_{(1)\mu\nu}, \tag{55}$$

it reduces to

$$\frac{d^2\xi^\alpha_{(1)}}{ds^2_{(0)}} + \partial_0 \Gamma^\alpha_{(1)\beta 0}\, \xi^\beta_{(0)} = \left(\partial_0 \Gamma^\alpha_{(1)\beta 0} - \partial_\beta \Gamma^\alpha_{(1)00}\right) \xi^\beta_{(0)}. \tag{56}$$

Canceling $\partial_0 \Gamma^\alpha_{(1)\beta 0}\, \xi^\beta_{(0)}$ on both sides, we get

$$\frac{d^2\xi^\alpha_{(1)}}{ds^2_{(0)}} = -\, \partial_\beta \Gamma^\alpha_{(1)00}\, \xi^\beta_{(0)}, \tag{57}$$

where

$$\Gamma^\alpha_{(1)00} = \frac{1}{2}\, \eta^{\alpha\rho} \left(2\, \partial_0 h_{(1)\rho 0} - \partial_\rho h_{(1)00}\right). \tag{58}$$

Specializing now to the transverse-traceless coordinate system, where the components $h_{(1)\rho 0}$ vanish identically, we obtain

$$\frac{d^2\xi^\alpha_{(1)}}{ds^2_{(0)}} = 0. \tag{59}$$

Without loss of generality, we can take the solution to be $\xi^{\alpha}_{(1)} = \text{constant}$. In the linear approximation, therefore, in consonance with the fact that linear gravitational waves do not transport energy nor momentum, particles are not affected by linear gravitational waves.[4]

4.3 Second-Order Effects

Up to second order, and already using the first order results, the geodesic deviation equation (48) reads

$$\frac{d^2\xi^{\alpha}_{(2)}}{ds^2_{(0)}} + \Gamma^{\alpha}_{(1)\gamma 0}\,\Gamma^{\gamma}_{(1)\beta 0}\,\xi^{\beta}_{(0)} + \partial_0\Gamma^{\alpha}_{(2)\beta 0}\,\xi^{\beta}_{(0)} = R^{\alpha}_{(2)00\beta}\,\xi^{\beta}_{(0)}. \tag{60}$$

Substituting the curvature tensor

$$R^{\alpha}_{(2)00\beta} = \partial_0\Gamma^{\alpha}_{(2)0\beta} - \partial_\beta\Gamma^{\alpha}_{(2)00} + \Gamma^{\alpha}_{(1)0\gamma}\Gamma^{\gamma}_{(1)0\beta} - \Gamma^{\alpha}_{(1)\beta\gamma}\Gamma^{\gamma}_{(1)00}, \tag{61}$$

and considering that in transverse-traceless coordinates $\Gamma^{\gamma}_{(1)00} = 0$, we obtain

$$\frac{d^2\xi^{\alpha}_{(2)}}{ds^2_{(0)}} = -\,\partial_\beta\Gamma^{\alpha}_{(2)00}\,\xi^{\beta}_{(0)}. \tag{62}$$

Now, in transverse-traceless coordinates, the second order Christoffel connection is

$$\Gamma^{\alpha}_{(2)00} = \partial_0 h^{\alpha}_{(2)0} - \frac{1}{2}\partial^{\alpha} h_{(2)00}. \tag{63}$$

The geodesic deviation equation reduces then to

$$\frac{d^2\xi^{\alpha}_{(2)}}{ds^2_{(0)}} = \left(\frac{1}{2}\partial_\beta\partial^{\alpha} h_{(2)00} - \partial_\beta\partial_0 h^{\alpha}_{(2)0}\right)\xi^{\beta}_{(0)}. \tag{64}$$

For definiteness, we consider a wave traveling in the z-direction, in which case k^{ρ} is given by (41). Let us then suppose two particles separated initially in the x-direction by a distance $\xi^{x}_{(0)}$, that is,

$$\xi^{\beta}_{(0)} = (0, \xi^{x}_{(0)}, 0, 0). \tag{65}$$

Considering that, in the proper frame $s_{(0)} = ct$, it is an easy task to verify that in this case the resulting equations of motion are

$$\frac{\partial^2\xi^{x}_{(2)}}{\partial t^2} = \frac{\partial^2\xi^{y}_{(2)}}{\partial t^2} = \frac{\partial^2\xi^{z}_{(2)}}{\partial t^2} = 0. \tag{66}$$

The same result is obtained for two particles separated initially in the y-direction. We consider now two particles separated initially in the z-direction by a distance $\xi^{z}_{(0)}$, that is,

$$\xi^{\beta}_{(0)} = (0, 0, 0, \xi^{z}_{(0)}). \tag{67}$$

In this case, the geodesic deviation equation (64) yields

$$\frac{\partial^2\xi^{x}_{(2)}}{\partial t^2} = \frac{\partial^2\xi^{y}_{(2)}}{\partial t^2} = 0\,, \tag{68}$$

[4] For a detailed discussion of this point, see [13].

but

$$\frac{1}{c^2}\frac{\partial^2 \xi^z_{(2)}}{\partial t^2} = \left(\partial_0 \partial_z h_{(2)z0} - \frac{1}{2}\partial_z \partial_z h_{(2)00}\right) \xi^z_{(0)}. \tag{69}$$

This means that a gravitational wave does not produce movement orthogonal to the direction of propagation. In other words, it is not an orthogonal, but a longitudinal wave. Notice that, in the second order, the two degrees of freedom are represented by $h_{(2)z0} = h_{(2)0z}$ and $h_{(2)00}$.

Considering that a detector on Earth will always be at large distances from the wave source, we use for $h_{(2)\mu\nu}$ the dominant solution (45). Furthermore, taking into account the arbitrariness of the wave vector K_ρ, we can choose it in such a way that $K_0 = K_1 = K_2 = 0$. In this case, the geodesic deviation equation (69) reduces to

$$\frac{\partial^2 \xi^z_{(2)}}{\partial t^2} = \xi^z_{(0)} \frac{\Phi_{(2)} z\, \omega^3}{4c} \sin[2\omega(t - z/c)]. \tag{70}$$

Although the wave amplitude decreases with distance, it can be assumed to be constant in the region of the experience. Accordingly, we write

$$\frac{\partial^2 \xi^z_{(2)}}{\partial t^2} = \frac{1}{4} \xi^z_{(0)} \Gamma_{(2)}\, \omega^2 \sin[2(\omega t - z/\lambdabar)], \tag{71}$$

where

$$\Gamma_{(2)} = \Phi_{(2)} \frac{z}{\lambdabar} \tag{72}$$

represents the wave amplitude at the region of the experience, with $\lambdabar = c/\omega$ the reduced wavelength.

Observe that now the origin of the coordinate z is completely arbitrary. We can then choose one of the particles to be at $z = 0$, in which case z will represent the position of the second particle. Assuming that the particles are initially ($t = 0$) at rest, the solution is found to be

$$\xi^z_{(2)} = -\frac{\xi^z_{(0)} \Gamma_{(2)}}{16} \Big[\sin[2(\omega t - z/\lambdabar)] - 2\omega t \cos[2z/\lambdabar] + \sin[2z/\lambdabar] \Big]. \tag{73}$$

For gravitational waves with wavelength much larger than the particle separation ($\lambdabar \gg z$), the solution becomes

$$\xi^z_{(2)} = -\frac{\xi^z_{(0)} \Gamma_{(2)}}{16} \Big[\sin(2\omega t) - 2\omega t \Big], \tag{74}$$

When a gravitational wave reaches two particles separated by a distance $\xi^z_{(0)}$ in the direction of the propagation, the distance between them will oscillate with frequency 2ω, and will grow linearly with time with a velocity

$$v = \frac{\xi^z_{(0)} \Gamma_{(2)}\, \omega}{8}. \tag{75}$$

This behavior is the result of tidal forces produced by the passage of a gravitational wave.

5 Final Remarks

Whenever use is made of a perturbation scheme, one forcibly ends up with a linear wave-equation. There is a widespread belief that gravitational waves can be approximately described by the solution of this linear wave equation. This assumption, however, is not justified. To understand it, let us make a comparison with gauge fields. As is well known, the gauge field of Chromodynamics must be nonlinear to transport color charge. Conversely, since electromagnetic waves are essentially linear, they are unable to transport their own source, that is, electric charge. Observe that, even though electromagnetic waves are linear, they do transport energy and momentum. This is possible because neither energy nor momentum is source of the electromagnetic field. As such, the energy-momentum current does not enter the electromagnetic field equation, and consequently its linearity does not restrict the energy-momentum current to be linear. Differently from electromagnetic waves, however, in order to transport energy and momentum (the source of gravitation), a gravitational wave must necessarily be nonlinear.

If we accept that the first-order equations are fully correct up to that order, and not just nearly correct as is usually assumed in the mixed approach, we arrive at the inexorable result that linear gravitational waves transport neither energy nor momentum. As a consequence, they are unable to produce any effect on free particles. One may wonder why the first order gravitational wave, which has a non-vanishing curvature tensor, produces no effects on free particles. To understand this question, lets consider the following points. First, because it enters the gravitational field equations, the energy-momentum density of any linear spacetime configuration must vanish because the energy-momentum current is at least quadratic in the field variables. A non-vanishing energy density can only appear in orders higher than one. This does not mean that the first-order gravitational field is physically meaningless. In fact, at the second order it will appear multiplied by itself, giving rise to nonlinear field configurations with non-vanishing energy-momentum density. Second, notice that the components of the Riemann tensor are not physically meaningful in the sense that they are different in different coordinate systems. For example, starting with the "electric components" R_{i0j0} of the Riemann tensor, through a general coordinate transformation one can get non-vanishing "magnetic components" R_{i0jk}. By inspecting the components, therefore, it is not possible to know whether they represent a true gravitomagnetic field, or just effects of coordinates. In order to get an answer, one needs to inspect the invariants constructed out of the Riemann tensor.[5] Now, as a simple calculation shows, *all invariants constructed out of the first-order Riemann tensor of the linear gravitational waves vanish identically* [17–19]. This includes the scalar curvature, the Kretschmann invariant, and the pseudo-scalar invariant. This means that the transverse components of the first-order wave are empty of physical meaning as no mass nor angular momentum (or helicity in the massless case) can be associated with them.

Motivated by the above results, we have then obtained the second-order solution to the gravitational field equations, which might represent a physical gravitational wave. Its amplitude depends explicitly on the frequency of the wave—a property typically related to nonlinearity. In contrast to the linear wave, the second-order wave is able to transport energy and momentum. This fact becomes evident if we notice that, at second order, the source energy-momentum tensor is conserved only in the covariant sense; namely, it is not really conserved [20]. This means that, differently from what happens at first order, at second order a mechanical system can lose energy in the form of gravitational waves. Furthermore, although the first-order field is transverse and traceless, the second order is longitudinal. This property can be understood by remembering that gravitational waves are generated, and act on particles, through tidal effects, which arise from inhomogeneities in the gravitational field. The effects they produce on free particles are

[5]For a discussion of this point, see [16, p. 355].

then found to consist of nonlinear oscillations along the direction of propagation. This is the signature a gravitational wave will leave in a detector, the effect to be looked for.

A crucial point of these waves refer to their frequency. As is well known, the quadrupole radiation emitted by a source propagates with twice the frequency of the source (this has to do with the tidal nature of the gravitational wave generation). However, the linear gravitational wave $h^{\mu\nu}_{(1)}$ emerges from the perturbation scheme propagating with the same frequency of the source. To circumvent this problem, one has to artificially adjust by hands the wave frequency (see, for example, [2, p. 105]). On the other hand, owing to its nonlinear nature, the second order gravitational wave $h^{\mu\nu}_{(2)}$ naturally emerges propagating with a frequency which is twice the source frequency, with the factor "2" coming from the fact that $h^{\mu\nu}_{(2)}$ depends quadratically on $h^{\mu\nu}_{(1)}$. This is in agreement with the quadrupole radiation property, as well as with the tidal nature of the gravitational radiation, and is a clear indication that it is not the first-order, but the second-order wave that represents the quadrupole (physical) gravitational wave. It is also important to observe that, due to an explicit additional linear dependence on the source distance r, the amplitude of the dominant part of $h^{\mu\nu}_{(2)}$ is found to fall off as $1/r$. Contrary to the usual belief, therefore, which presupposes that second-order effects fall off as $1/r^2$, second-order effects are not necessarily neglectful at large distances from the source.

It is important to remark finally that, according to Birkhoff's theorem,[6] any spherical source produces a time-independent gravitational field outside it. As a consequence, no spherically symmetric longitudinal gravitational waves can exist. However, due to the explicit dependence of the amplitude coefficient (35) on the wave number, we see that the nonlinear gravitational wave considered here will never be spherically symmetric. The usual restrictions imposed by Birkhoff's theorem on longitudinal gravitational waves, therefore, do not apply to the present case of longitudinal gravitational waves.

Acknowledgements The authors would like to thank FAPESP, CNPq and CAPES for partial financial support.

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[6] For a textbook reference, see [21].

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